

VECTOR-VALUED EIDELHEIT SEQUENCES AND THE NON B-COMPLETENESS OF TENSOR PRODUCTS

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ABSTRACT. We introduce vector-valued Eidelheit sequences and obtain a characterization that generalizes Eidelheit's classical theorem [5]. As an application, we discuss criteria for non B -completeness of completed tensor products.

1. INTRODUCTION

In 1936 Eidelheit [5] (see also [11, Thm. 26.27]) characterized the sequences $(x'_n)_{n \in \mathbb{N}}$ of continuous linear functionals on a Fréchet space E such that for each arbitrary sequence $(a_n)_{n \in \mathbb{N}}$ of complex numbers the infinite system of linear equations

$$(1.1) \quad \langle x'_n, x \rangle = a_n, \quad n \in \mathbb{N},$$

has solutions $x \in E$. In his memory such sequences are nowadays called *Eidelheit sequences*. Later on, this notion has been extensively studied by Mityagin [12] and Vogt [24, 25, 26].

In this article we shall study the corresponding *vector-valued* Eidelheit problem. Our results reveal an interesting connection between Eidelheit sequences and Pták's notion of B -completeness. Let E be a Hausdorff locally convex space. For each $n \in \mathbb{N}$ consider a continuous linear mapping T_n from E into a Hausdorff locally convex space F_n . We write $T_n(x) = \langle T_n, x \rangle$. Suppose that E is B -complete and each F_n is barreled. In our main theorem we shall give necessary and sufficient conditions over the linear mappings $(T_n)_{n \in \mathbb{N}}$ such that for any arbitrary choice of $y_n \in F_n$, $n \in \mathbb{N}$, the infinite system of linear equations

$$(1.2) \quad \langle T_n, x \rangle = y_n, \quad n \in \mathbb{N},$$

admits solutions $x \in E$. In the scalar-valued case ($F = \mathbb{C}$) our conditions are equivalent to those in Eidelheit's theorem. Hence, we extend here his characterization to B -complete spaces.

As a rather surprising application, we shall also provide criteria on a Fréchet space and a reflexive (DF) -space which ensure that their completed tensor product (with respect to the ε -, projective, biequicontinuous, or inductive topology) is *not* B -complete. The latter is partly based on the work of Valdivia [21]. As a corollary, we show that the space $\mathcal{O}_C(\mathbb{R}^d)$ of very slowly increasing functions [8] is not B -complete.

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2. PRELIMINARIES

Throughout this article every locally convex space (from now on abbreviated as l.c.s.) is assumed to be Hausdorff. Whenever we want to specify a topology on its topological dual E' we will employ the subscript σ for the weak* topology, τ for the Mackey dual topology, or b for the strong topology. The polar in E' of a set $U \subseteq E$ is denoted by U° . For a l.c.s. F , we write $L(E, F)$ for the space of all continuous linear mappings from E into F . As in the introduction, we often write $T(x) = \langle T, x \rangle$, where $T \in L(E, F)$. Cartesian products and direct sums of l.c.s. are topologized in the standard fashion [9].

One of the central notions in this article is the one of B -completeness [8, Chap. 3, Sect. 17], [2, Chap. 7]. A l.c.s. E is said to be B -complete (or a Pták space) if a linear subspace $X \subseteq E'$ is closed in E'_σ if and only if $X \cap A$ is closed in A (endowed with the relative topology of E'_σ) for every equicontinuous set $A \subset E'$. Every Fréchet space is B -complete by the Krein-Šmulian theorem [8, Chap. 3, Thm. 10.2]. Moreover, every reflexive (DF) -space is B -complete [8, Chap. 3, Prop. 17.6].

Concerning topological tensor products we shall use the following terminology [2, 6, 16]. Let E and F be locally convex spaces. The tensor product $E \otimes F$ is canonically isomorphic to the space $B(E'_\sigma, F'_\sigma)$ of jointly continuous bilinear functionals on $E'_\sigma \times F'_\sigma$ via

$$(x \otimes y)(x', y') = \langle x', x \rangle \langle y', y \rangle, \quad x' \in E', \ y' \in F'.$$

The ε -topology on $E \otimes F$ is defined as the topology carried over from $B(E'_\sigma, F'_\sigma)$ endowed with the topology of uniform convergence on the products of equicontinuous subsets of E' and F' . The π - (resp., β -, i -) topology on $E \otimes F$ is the finest locally convex topology on this vector space for which the canonical bilinear mapping $E \times F \rightarrow E \otimes F$ is jointly (resp., hypo-, separately) continuous. Let $t = \varepsilon, \pi, \beta$ or i . Endowed with the t -topology, the space $E \otimes F$ will be denoted by $E \otimes_t F$ and its completion by $E \widehat{\otimes}_t F$. Let G and H be l.c.s. and let $T : E \rightarrow G$, $S : F \rightarrow H$ be continuous linear mappings. The associated tensor mapping $T \otimes S : E \otimes_t F \rightarrow G \otimes_t H$ is continuous. We write $S \widehat{\otimes}_t T$ for the extension of $S \otimes T$ as a continuous linear mapping from $E \widehat{\otimes}_t F$ into $G \widehat{\otimes}_t H$.

3. VECTOR-VALUED EIDELHEIT SEQUENCES

Let E be a locally convex space. For each $n \in \mathbb{N}$, we consider $T_n \in L(E, F_n)$, where F_n is a locally convex space. We seek conditions over $(T_n)_n$ such that for each $(y_n)_n \in \prod_{n \in \mathbb{N}} F_n$ the infinite system of equations (1.2) admits solutions $x \in E$. If this is the case, we call $(T_n)_n$ an *Eidelheit sequence* (or, in short, *Eidelheit*). To the system of equations (1.2) we associate the mapping

$$\Lambda : E \rightarrow \prod_{n \in \mathbb{N}} F_n : x \rightarrow (\langle T_n, x \rangle)_n.$$

Naturally, $(T_n)_n$ is Eidelheit if and only if Λ is surjective.

Theorem 3.1. *Let E be a l.c.s., $(F_n)_n$ be a sequence of l.c.s., and $(T_n)_n \in \prod_{n \in \mathbb{N}} L(E, F_n)$. If $(T_n)_n$ is Eidelheit, then the following three properties hold:*

(P1) For every $N \in \mathbb{N}$ and $y'_0 \in F'_0, \dots, y'_N \in F'_N$

$$\sum_{n=0}^N y'_n \circ T_n = 0$$

implies $y'_0 = \dots = y'_N = 0$.

(P2) For every equicontinuous set $A \subset E'$ there is $\nu \in \mathbb{N}$ such that for every $N > \nu$ and $y'_0 \in F'_0, \dots, y'_N \in F'_N$

$$\sum_{n=0}^N y'_n \circ T_n \in A$$

implies $y'_{\nu+1} = \dots = y'_N = 0$.

(P3) For every $N \in \mathbb{N}$ the mapping

$$E \rightarrow \prod_{n=0}^N F_n : x \rightarrow (\langle T_0, x \rangle, \dots, \langle T_N, x \rangle)$$

is surjective.

Conversely, suppose that E is B -complete and each F_n is barreled. If $(T_n)_n$ satisfies (P1), (P2), and (P3), then $(T_n)_n$ is Eidelheit.

Proof. The transpose of Λ is given by

$$\Lambda^t := \bigoplus_{n \in \mathbb{N}} F'_n \rightarrow E' : (y'_0, \dots, y'_N) \rightarrow \sum_{n=0}^N y'_n \circ T_n = \sum_{n=0}^N T_n^t(y'_n).$$

Suppose that $(T_n)_n$ is Eidelheit. The property (P3) trivially holds. Since Λ is continuous and surjective, Λ^t is a weak*-topological isomorphism onto its image. In particular, (P1) holds. Next, we actually show that a stronger property than (P3) holds, namely, that it is satisfied not only for equicontinuous sets but also for any bounded subset of E'_σ . So, let A be a bounded set in E'_σ . The set $B = (\Lambda^t)^{-1}(A)$ is weakly* bounded, and thus bounded in the Mackey dual topology, as follows from Mackey's theorem [9, p. 254]. We have that $(\prod_{n \in \mathbb{N}} F_n)'_\tau = \bigoplus_{n \in \mathbb{N}} (F_n)'_\tau$ [9, p. 286]. Hence $B \subset \bigoplus_{n=0}^\mu F'_n$ for some $\mu \in \mathbb{N}$ [9, p. 213].

Conversely, suppose that E is B -complete, each F_n is barreled, and that $(T_n)_n$ satisfies (P1), (P2), and (P3). From (P1) and the Hahn-Banach theorem we conclude that $\Lambda(E)$ is dense in $\prod_{n \in \mathbb{N}} F_n$ and therefore we only need to show that $\Lambda(E)$ is closed in $\prod_{n \in \mathbb{N}} F_n$. Since the quotient of a B -complete space with a closed subspace is again B -complete [8, Chap. 3, Prop. 17.2] and B -complete spaces are always complete [8, Prop. 17.3(b)], it suffices to show that Λ is a topological homomorphism. Hence, in view of [16, Lemma 37.7], we need to show that (i) $\Lambda^t(\bigoplus_{n \in \mathbb{N}} F'_n)$ is closed in E'_σ and that (ii) for every equicontinuous set $A \subset E'$ there is an equicontinuous set $B \subset \bigoplus_{n \in \mathbb{N}} F'_n$ such that $\Lambda^t(\bigoplus_{n \in \mathbb{N}} F'_n) \cap A \subseteq \Lambda^t(B)$. We first show (i). Since E is B -complete, we have to show that $\Lambda^t(\bigoplus_{n \in \mathbb{N}} F'_n) \cap A$ is closed in A for every equicontinuous set $A \subset E'$.

By **(P2)** there is $\nu \in \mathbb{N}$ such that $\Lambda^t(\bigoplus_{n \in \mathbb{N}} F'_n) \cap A = \Lambda_\nu^t(\bigoplus_{n=0}^\nu F'_n) \cap A$, where

$$\Lambda_\nu : E \rightarrow \prod_{n=0}^\nu F_n : x \rightarrow (\langle T_0, x \rangle, \dots, \langle T_\nu, x \rangle).$$

The property **(P3)** says that the mapping Λ_ν is surjective, the Pták open mapping theorem [8, Chap. 3, Prop. 17.2] then implies that Λ_ν is a topological homomorphism. Consequently, $\Lambda_\nu^t(\bigoplus_{n=0}^\nu F'_n)$ is closed in E'_σ , which yields the assertion. Finally, let us verify (ii). Fix an equicontinuous subset A of E' . By **(P2)** there is $\nu \in \mathbb{N}$ such that $\Lambda^t(\bigoplus_{n \in \mathbb{N}} F'_n) \cap A = \Lambda_\nu^t(\bigoplus_{n=0}^\nu F'_n) \cap A$. Since Λ_ν is a topological homomorphism, there is an equicontinuous subset B of $\bigoplus_{n=0}^\nu F'_n$, and thus a fortiori of $\bigoplus_{n \in \mathbb{N}} F'_n$, such that $\Lambda_\nu^t(\bigoplus_{n=0}^\nu F'_n) \cap A \subseteq \Lambda_\nu^t(B) = \Lambda^t(B)$. \square

Setting $F_n = \mathbb{C}$, $n \in \mathbb{N}$, in Theorem 3.1 we obtain the following criterion for scalar-valued Eidelheit sequences:

Corollary 3.2. *Let E be a l.c.s. and let $(x'_n)_n \subset E'$. If $(x'_n)_n$ is Eidelheit, then:*

- (P1) *The set of linear functionals $\{x'_n : n \in \mathbb{N}\}$ is linearly independent.*
- (P2) *For every equicontinuous set $A \subset E'$ the set*

$$\text{span}\{x'_n : n \in \mathbb{N}\} \cap A$$

is contained in a finite-dimensional subspace.

Conversely, if E is B -complete, the properties (P1) and (P2) are also sufficient for $(x'_n)_n$ to be an Eidelheit sequence.

Proof. Indeed, property **(P3)** is superfluous for $F_n = \mathbb{C}$, $n \in \mathbb{N}$, since it is always implied by (P1) and the Hahn-Banach theorem. \square

When E is a Fréchet space, we recover Eidelheit's original theorem [5].

4. ASSOCIATED VECTOR-VALUED EIDELHEIT SEQUENCES

We can canonically associate a vector-valued system of equations to each scalar-valued system of equations (1.1) in the following way: Let E and F be l.c.s. and suppose that F is complete. For a given sequence $(x'_n)_n \subset E'$ we call $(x'_n \widehat{\otimes}_t \text{Id}_F)_n \subset L(E \widehat{\otimes}_t F, F)$, with $t = \varepsilon, \pi, \beta$, or i , the F -associated sequence of (x'_n) (with respect to the t -topology).

In this section we discuss whether the F -associated sequence of an Eidelheit sequence is itself Eidelheit. We start by giving sufficient conditions on E and F such that the F -associated sequence of any Eidelheit sequence in E' is again Eidelheit.

Proposition 4.1. *Let E and F be locally convex spaces.*

- (i) *Let $t = \varepsilon$ or π . If E is B -complete, F is barreled and complete, and $E \widehat{\otimes}_t F$ is B -complete, then for any Eidelheit sequence $(x'_n)_n \subset E'$ its F -associated sequence, with respect to the t -topology, is Eidelheit.*
- (ii) *If E is B -complete, F is barreled complete and contains a bounded total set, and $E \widehat{\otimes}_\beta F$ is B -complete, then for any Eidelheit sequence $(x'_n)_n \subset E'$ its F -associated sequence, with respect to the β -topology, is Eidelheit.*

Proof. It suffices to show that $(x'_n \widehat{\otimes}_t \text{Id}_F)_n$ satisfies condition **(P1)**, **(P2)**, and **(P3)** of Theorem 3.1 for $t = \varepsilon, \pi$, or β , under the corresponding hypotheses. Notice that

$$y' \circ (x' \widehat{\otimes}_t \text{Id}_F) = x' \widehat{\otimes}_t y', \quad x' \in E', \ y' \in F'.$$

We simultaneously show that **(P1)** and **(P3)** must hold, a case distinction is only necessary in the proof of **(P2)**.

(P1): Let $y'_0, \dots, y'_N \in F'$ be such that

$$\sum_{n=0}^N x'_n \widehat{\otimes}_t y'_n = 0.$$

Since the set $\{x'_n : n \in \mathbb{N}\}$ is linearly independent, we obtain that $y'_n = 0$ for all $n = 0, \dots, N$.

(P3): Let $y_0, \dots, y_N \in F$ be arbitrary. Since the set of vectors $\{x'_n : n \in \mathbb{N}\}$ is linearly independent, there are elements $x_0, \dots, x_N \in E$ such that

$$\langle x'_n, x_k \rangle = \delta_{k,n}, \quad k, n = 0, \dots, N,$$

where $\delta_{k,n}$ is the Kronecker delta. Hence,

$$\langle x'_n \widehat{\otimes}_t \text{Id}_F, \sum_{k=0}^N x_k \otimes y_k \rangle = \sum_{k=0}^N \langle x'_n, x_k \rangle y_k = y_n, \quad n = 0, \dots, N.$$

(P2): We now treat the cases (i) and (ii) separately.

(i) Let $t = \varepsilon$ or π . Since the canonical bilinear mapping $E \times F \rightarrow E \otimes_t F$ is jointly continuous, it suffices to show that for arbitrary open neighborhoods of the origin U and V in E and F , respectively, there is $\nu \in \mathbb{N}$ such that for all $N > \nu$ and $y'_0, \dots, y'_N \in F'$ it holds that

$$(4.1) \quad \sum_{n=0}^N x'_n \widehat{\otimes}_t y'_n \in (U \otimes V)^\circ$$

implies $y'_{\nu+1} = \dots = y'_N = 0$. Let $\nu \in \mathbb{N}$ be such that for all $N > \nu$ and $b_0, \dots, b_N \in \mathbb{C}$

$$(4.2) \quad \sum_{n=0}^N b_n x'_n \in U^\circ$$

implies $b_{\nu+1} = \dots = b_N = 0$. Now suppose that (4.1) holds. We have

$$\sum_{n=0}^N \langle y'_n, y \rangle x'_n \in U^\circ$$

for all $y \in V$. Property (4.2) and the fact that V is absorbing therefore imply that $y'_{\nu+1} = \dots = y'_N = 0$.

(ii) We now consider $t = \beta$. Let B be a bounded total set in F . Since the canonical bilinear mapping $E \times F \rightarrow E \otimes_\beta F$ is hypocontinuous, it suffices to show that for an

arbitrary open neighborhood of the origin U in E there is $\nu \in \mathbb{N}$ such that for all $N > \nu$ and $y'_0, \dots, y'_N \in F$ it holds that

$$(4.3) \quad \sum_{n=0}^N x'_n \widehat{\otimes}_\beta y'_n \in (U \otimes B)^\circ$$

implies $y'_{\nu+1} = \dots = y'_N = 0$. Let $\nu \in \mathbb{N}$ be such that for all $N > \nu$ and $b_0, \dots, b_N \in \mathbb{C}$ it holds that

$$(4.4) \quad \sum_{n=0}^N b_n x'_n \in U^\circ$$

implies $b_{\nu+1} = \dots = b_N = 0$. Now suppose that (4.3) holds. We have

$$\sum_{n=0}^N \langle y'_n, y \rangle x'_n \in U^\circ$$

for all $y \in B$. Property (4.4) and the fact that B is total therefore imply that $y'_{\nu+1} = \dots = y'_N = 0$. \square

Corollary 4.2. *Let E and F be Fréchet spaces. For any Eidelheit sequence $(x'_n)_n \subset E'$ its F -associated sequence, with respect to both the ε - and π -topology, is also Eidelheit.*

Remark 4.3. For the π -topology, Corollary 4.2 can also be shown in the following alternative way: We need to show that the mapping

$$\Lambda_F : E \widehat{\otimes}_\pi F \rightarrow F^\mathbb{N} : \Phi \rightarrow (\langle x'_n \widehat{\otimes}_\pi \text{Id}_F, \Phi \rangle)_n$$

is surjective. Notice that $\Lambda_F = \Lambda \widehat{\otimes}_\pi \text{Id}_F$, where

$$\Lambda : E \rightarrow \mathbb{C}^\mathbb{N} : x \rightarrow (\langle x'_n, x \rangle)_n.$$

The surjectivity of Λ_F then immediately follows from the ensuing well known fact [16]: Given two surjective continuous linear mappings $T_1 : E_1 \rightarrow F_1$ and $T_2 : E_2 \rightarrow F_2$ between Fréchet spaces, the mapping

$$T_1 \widehat{\otimes}_\pi T_2 : E_1 \widehat{\otimes}_\pi E_2 \rightarrow F_1 \widehat{\otimes}_\pi F_2$$

is also surjective.

Remark 4.4. Since B -completeness is hard to verify except when implied by formally stronger properties, in practice only Corollary 4.2 seems to be directly applicable. However, for many classical Eidelheit type problems (e.g. the Borel problem), the (DF) -spaces F for which the associated F -valued problem is solvable have been characterized by dual (DN) - (Ω) type conditions; see [22] for the Borel problem, [3] for holomorphic interpolation, and [1] for real analytic interpolation. On the other hand, Proposition 4.1 is a useful device for proving non B -completeness, as we show in the rest of the article.

Next, we are interested in giving sufficient conditions on E and F such that the F -associated sequence of any sequence in E' is *not* Eidelheit. The idea is to find conditions on E and F ensuring that their completed tensor product is “small”. Let us

fix some terminology. A l.c.s. F is called a (LF) -space (resp., (LB) -space) if F is the locally convex inductive limit of an inductive sequence $(F_n)_{n \in \mathbb{N}}$ of Fréchet spaces (resp., Banach spaces). The sequence $(F_n)_n$ is called *strict* if F_n is a topological subspace of F_{n+1} for each $n \in \mathbb{N}$. In such a case, we also say that F is strict. It is well known that every strict (LF) -space F is complete, *regular*, i.e., every bounded set in F is contained and bounded in some step F_N , and that, for every $n \in \mathbb{N}$, the induced topology of F on F_n coincides with the original topology on F_n . If $F_n \subsetneq F_{n+1}$ for each $n \in \mathbb{N}$ we say that F is *proper*. Finally, we will write E'_A for the linear span of a subset $A \subseteq E'$.

Lemma 4.5. *Let E be a Fréchet space. Then, E is non-normable if and only if E' contains an Eidelheit sequence.*

Proof. Suppose E is non-normable. Let $(U_n)_{n \in \mathbb{N}}$ be a fundamental system of neighborhoods of the origin in E consisting of closed convex balanced sets such that $U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$. The non-normability of E and the bipolar theorem imply that there is a sequence of natural numbers $(k_n)_n$ such that $E'_{U_{k_n}^\circ} \subsetneq E'_{U_{k_{n+1}}^\circ}$ for each $n \in \mathbb{N}$. Choose $x'_0 \in E'_{U_{k_0}^\circ}$, $x'_0 \neq 0$, and $x'_n \in E'_{U_{k_n}^\circ} \setminus E'_{U_{k_{n-1}}^\circ}$, $n \in \mathbb{Z}_+$. Clearly, the sequence $(x'_n)_n$ satisfies the conditions of Corollary 3.2. Conversely, no sequence in a Banach space can be Eidelheit, as follows from Corollary 3.2. \square

Proposition 4.6.

- (i) *Let E be a complete semi-reflexive nuclear l.c.s. that admits a continuous norm and let $F = \varinjlim_n F_n$ be a proper strict (LF) -space. Then, for any sequence in E' its F -associated sequence, with respect to both the ε - and π -topology, is not Eidelheit.*
- (ii) *Let G be a complete semi-reflexive nuclear l.c.s., H a non-normable reflexive Fréchet space, and suppose that $G \widehat{\otimes}_\pi H$ is bornological. Set $E = G'_b$ and $F = H'_b$. Then, for any sequence in E' its F -associated sequence, with respect to ε -topology, is not Eidelheit.*

Proof. (i) Since E is nuclear, there is no distinction between the ε - and π -topology and we simply write $E \widehat{\otimes} F \cong E \widehat{\otimes}_\varepsilon F \cong E \widehat{\otimes}_\pi F$. Notice that, since E is complete and nuclear, and F complete, we have that $E \widehat{\otimes} F \cong L(E'_b, F)$. Let $(x'_n)_n \subset E'$ be arbitrary. Choose $y_0 \in F_0$ and $y_n \in F_n \setminus F_{n-1}$, $n \in \mathbb{Z}_+$, and suppose that there is $\Phi \in L(E'_b, F)$ such that

$$(4.5) \quad y_n = x'_n \widehat{\otimes} \text{Id}_F(\Phi) = \Phi(x'_n), \quad n \in \mathbb{N}.$$

The fact that E is semi-reflexive and admits a continuous norm implies that E'_b contains a bounded total set B . As strict inductive limits are regular, we have that $\Phi(B) \subset F_N$ for some $N \in \mathbb{N}$. Since F_N is closed in F , we obtain that

$$\Phi(E') = \Phi(\overline{E'_B}) \subseteq \overline{\Phi(E'_B)}^F \subseteq \overline{F_N}^F = F_N.$$

In view of the choice of the sequence $(y_n)_n$ this contradicts (4.5).

(ii) By [6, Chap. II, Corollaire, p. 90] we have that

$$G'_b \widehat{\otimes}_i H'_b \cong (G \widehat{\otimes}_\pi H)' \cong B(G, H),$$

where $B(G, H)$ denotes the space of jointly continuous bilinear forms on $G \times H$. Let $(g_n)_n \subset E' = G$ be arbitrary. Lemma 4.5 implies that $F = H'$ contains an Eidelheit sequence $(h'_n)_n$. Suppose that there is $\Phi \in B(G, H)$ such that

$$h'_n = g_n \widehat{\otimes}_i \text{Id}_{H'}(\Phi) = \Phi(g_n, \cdot), \quad n \in \mathbb{N}.$$

Since the mapping Φ is jointly continuous, there are neighborhoods of the origin U and V of E and F , respectively, such that

$$|\Phi(g, h)| \leq 1, \quad g \in U, h \in V.$$

Hence,

$$h'_n = \Phi(g_n, \cdot) \in \bigcup_{\lambda > 0} \lambda V^\circ, \quad n \in \mathbb{N}.$$

Since we have chosen $(h'_n)_n$ to be Eidelheit, this contradicts property (P2) of Corollary 3.2. \square

5. APPLICATION: NON B -COMPLETENESS OF COMPLETED TENSOR PRODUCTS

In this section we give sufficient conditions on a Fréchet space E and a reflexive (DF) -space F for $E \widehat{\otimes}_t F$, with $t = \varepsilon, \pi, \beta$, or i , not to be B -complete. The results are based on the work of Valdivia [21] and our results from Section 4. We mention that the question whether the β -completed tensor product of two B -complete spaces is again B -complete was first raised by Summers [15].

Theorem 5.1. *Let E and F be locally convex spaces.*

- (i) *The spaces $E \widehat{\otimes}_\varepsilon F$ and $E \widehat{\otimes}_\pi F$ are not B -complete in the following cases:*
 - (a) *E is an infinite-dimensional nuclear Fréchet space that admits a continuous norm and F is a reflexive proper strict (LB) -space.*
 - (b) *E is a Fréchet space which does not admit a continuous norm and F is an infinite-dimensional Montel (DF) -space such that $F \not\cong \mathbb{C}^{(\mathbb{N})}$.*
- (ii) *The spaces $E \widehat{\otimes}_\beta F$ and $E \widehat{\otimes}_i F$ are not B -complete in the following cases:*
 - (a) *E is an infinite-dimensional nuclear Fréchet space, F is a non-normable reflexive (DF) -space that contains a bounded total set, and $E'_b \widehat{\otimes}_\pi F'_b$ is bornological.*
 - (b) *E is an infinite-dimensional Fréchet-Montel space such that $E \not\cong \mathbb{C}^{(\mathbb{N})}$ and F is a reflexive (DF) -space that does not contain a bounded total set.*

On the other hand, in all four cases, the spaces E and F are B -complete.

In the proof of Theorem 5.1 we shall employ the following result of Valdivia. Given a l.c.s. E we employ the notation

$$E^{\mathbb{N}} = \prod_{n \in \mathbb{N}} E, \quad E^{(\mathbb{N})} = \bigoplus_{n \in \mathbb{N}} E.$$

Proposition 5.2. [21, Thm. 6] *Let E be an infinite-dimensional Fréchet-Montel space. The following facts are equivalent:*

- (i) *$E^{(\mathbb{N})}$ is B -complete,*
- (ii) *$(E'_b)^{\mathbb{N}}$ is B -complete,*

(iii) $E \cong \mathbb{C}^{\mathbb{N}}$.

Proof of Theorem 5.1. (i) Case (a): It follows directly from Lemma 4.5 and Propositions 4.1(i) and 4.6(i). Case (b): Let $t = \varepsilon$ or π . By [6, Chap. II, Lemme 10.2, p. 93] the space E contains a complemented subspace that is isomorphic to $\mathbb{C}^{\mathbb{N}}$. Hence the space $E \widehat{\otimes}_t F$ contains a complemented subspace that is isomorphic to $\mathbb{C}^{\mathbb{N}} \widehat{\otimes}_t F$. As closed subspaces of B -complete spaces are B -complete [8, Chap. 3, Prop. 17.4], it suffices to show that $\mathbb{C}^{\mathbb{N}} \widehat{\otimes}_t F$ is not B -complete. Set $G = F'_b$, an infinite-dimensional Fréchet-Montel space such that $G \not\cong \mathbb{C}^{\mathbb{N}}$. Since,

$$\mathbb{C}^{\mathbb{N}} \widehat{\otimes}_t F \cong F^{\mathbb{N}} \cong (G'_b)^{\mathbb{N}}$$

as l.c.s., this follows from Proposition 5.2.

(ii) Since, in both cases, the spaces E and F are barreled, we have that $E \widehat{\otimes}_\beta F = E \widehat{\otimes}_i F$. Therefore we shall simply denote this space by $E \bar{\otimes} F$. Case (a): It follows directly from Lemma 4.5 and Propositions 4.1(ii) and 4.6(ii). Case (b): By [6, Chap. II, Lemme 10.1, p. 93] the space F contains a complemented subspace that is isomorphic to $\mathbb{C}^{(\mathbb{N})}$. Hence the space $E \bar{\otimes} F$ contains a complemented subspace that is isomorphic to $E \bar{\otimes} \mathbb{C}^{(\mathbb{N})}$. As closed subspaces of B -complete spaces are B -complete, it suffices to show that $E \bar{\otimes} \mathbb{C}^{(\mathbb{N})}$ is not B -complete. But

$$E \bar{\otimes} \mathbb{C}^{(\mathbb{N})} \cong E^{(\mathbb{N})}$$

as l.c.s., so that the result follows from Proposition 5.2. \square

As an application of Theorem 5.1, we now show that the space

$$\mathcal{O}_C(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \exists N \in \mathbb{N} \forall \alpha \in \mathbb{N}^d : \sup_{x \in \mathbb{R}^d} (1 + |x|)^{-N} |f^{(\alpha)}(x)| < \infty\}$$

of very slowly increasing functions endowed with its canonical (LF) -topology is not B -complete. We denote by s the space of rapidly decreasing sequences.

Proposition 5.3. *The space $\mathcal{O}_C(\mathbb{R}^d)$ is not B -complete.*

Proof. Valdivia [20] has shown that

$$\mathcal{O}_M(\mathbb{R}^d) \cong s \widehat{\otimes}_\pi s',$$

as l.c.s., where $\mathcal{O}_M(\mathbb{R}^d)$ denotes the space of slowly increasing functions. Since $\mathcal{O}'_M(\mathbb{R}^d) \cong \mathcal{O}_C(\mathbb{R}^d)$, as l.c.s., via the Fourier transform and the space $s \widehat{\otimes}_\pi s'$ is bornological [6, Chap. II, Corollaire 2, p. 128] (see [10] for a modern proof using homological algebra techniques), [6, Chap. II, Corollaire, p. 90] implies that

$$\mathcal{O}_C(\mathbb{R}^d) \cong s' \widehat{\otimes}_i s.$$

Hence $\mathcal{O}_C(\mathbb{R}^d)$ is not B -complete by Theorem 5.1(ii)(a). \square

We end this article by giving an overview of the lack of B -completeness of various spaces occurring in the theory of (ultra)distributions. Concerning ultradistributions, we follow the Braun-Meise-Taylor approach [4]. Since we shall employ sequential representations, we need to introduce power series spaces [11, Chap. 29]. Let $\alpha = (\alpha_n)_{n \in \mathbb{N}}$

be a sequence of positive real numbers tending monotonically increasing to infinity. For $h \in \mathbb{R}$ we set

$$\Lambda_h^\alpha = \{(c_n)_n \in \mathbb{C}^\mathbb{N} : |(c_n)_n|_h := \sup_{n \in \mathbb{N}} |c_n| e^{h\alpha_n} < \infty\}.$$

Define the Fréchet spaces

$$\Lambda_0(\alpha) = \varprojlim_{h \rightarrow 0^+} \Lambda_{-h}^\alpha, \quad \Lambda_\infty(\alpha) = \varprojlim_{h \rightarrow \infty} \Lambda_h^\alpha.$$

The space $\Lambda_0(\alpha)$ (resp., $\Lambda_\infty(\alpha)$) is nuclear [11, Prop. 29.6] if

$$\lim_{n \rightarrow \infty} \frac{\log n}{\alpha_n} = 0 \quad \left(\text{resp., } \sup_{n \in \mathbb{N}} \frac{\log n}{\alpha_n} < \infty \right).$$

Their strong duals are given by

$$\Lambda'_0(\alpha) = \varinjlim_{h \rightarrow 0^+} \Lambda_h^\alpha, \quad \Lambda'_\infty(\alpha) = \varinjlim_{h \rightarrow \infty} \Lambda_{-h}^\alpha.$$

Proposition 5.4. *Let $\Omega \subseteq \mathbb{R}^d$ be an open connected set and let ω be a non-quasianalytic weight function in the sense of [4]. The spaces $\mathcal{D}(\Omega)$, $\mathcal{D}'(\Omega)$, $\mathcal{D}_{(\omega)}(\Omega)$, $\mathcal{D}'_{(\omega)}(\Omega)$, $\mathcal{E}_{\{\omega\}}(\Omega)$, and $\mathcal{E}'_{\{\omega\}}(\Omega)$ are not B -complete.*

Proof. Set $\alpha_n = \omega(n^{1/d})$, $n \in \mathbb{N}$. The assertion is a direct consequence of Theorem 5.1 and the ensuing sequential representations due to Valdivia and Vogt [19, 23]:

$$\begin{aligned} \mathcal{D}(\Omega) &\cong s\widehat{\otimes}_i \mathbb{C}^\mathbb{N}, & \mathcal{D}'(\Omega) &\cong s'\widehat{\otimes}_\pi \mathbb{C}^\mathbb{N}, \\ \mathcal{D}_{(\omega)}(\Omega) &\cong \Lambda_\infty(\alpha)\widehat{\otimes}_i \mathbb{C}^\mathbb{N}, & \mathcal{D}'_{(\omega)}(\Omega) &\cong \Lambda'_\infty(\alpha)\widehat{\otimes}_\pi \mathbb{C}^\mathbb{N}, \\ \mathcal{E}_{\{\omega\}}(\Omega) &\cong \Lambda'_0(\alpha)\widehat{\otimes}_\pi \mathbb{C}^\mathbb{N}, & \mathcal{E}'_{\{\omega\}}(\Omega) &\cong \Lambda_0(\alpha)\widehat{\otimes}_i \mathbb{C}^\mathbb{N}. \end{aligned}$$

□

Remark 5.5. The question whether the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are B -complete was first posed by Raikov and attracted the attention of many authors [7, 13, 14, 17, 18]. The non B -completeness of $\mathcal{D}_{(\omega)}(\Omega)$ and $\mathcal{D}'_{(\omega)}(\Omega)$ is due to Valdivia [21]. Furthermore, the lack of B -completeness of $\mathcal{E}_{\{\omega\}}(\Omega)$ and $\mathcal{E}'_{\{\omega\}}(\Omega)$ follows directly from their sequential representations and Proposition 5.2, and is therefore implicitly due to Valdivia and Vogt. However, the fact that the space $\mathcal{O}_C(\mathbb{R}^d)$ is not B -complete seems to be new and does not follow from Valdivia's work.

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